Unit 5: Special Problems in Mechanics

The "special problems" we will address in this chapter deal with four common mechanical systems: pulleys, inclined planes, springs, and pendulums. These systems pop up in many mechanics problems, and it will save you time and points if you familiarize yourself with their quirks. These systems obey the same mechanical rules as the rest of the world, and we will only introduce one principle (Hooke's Law) that hasn't been covered in the previous three units. However, there are a number of problem-solving techniques that are particular to these sorts of problems, and mastering them will help you get through these problems quickly and easily.

The Three-Step Approach to Problem Solving

The systems we will look at in this chapter won't test your knowledge of obscure formulas so much as your problem-solving abilities. The actual physics at work on these systems is generally quite simple—it rarely extends beyond Newton's three laws and a basic understanding of work and energy—but you'll need to apply this simple physics in imaginative ways.

There are three general steps you can take when approaching any problem in mechanics. Often the problems are simple enough that these steps are unnecessary. However, with the special problems we will tackle in this chapter, following these steps carefully may save you many times over in physics. The three steps are:

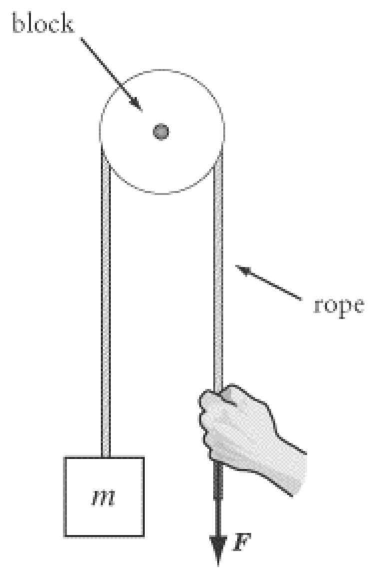
1. **Ask yourself how the system will move:** Before you start writing down equations and looking at answer choices, you should develop an intuitive sense of what you're looking at. In what direction will the objects in the system move? Will they move at all? Once you know what you're dealing with, you'll have an easier time figuring out how to approach the problem.

2. **Choose a coordinate system:** Most systems will only move in one dimension: up and down, left and right, or on an angle in the case of inclined planes. Choose a coordinate system where one direction is negative, the other direction is positive, and, if necessary, choose an origin point that you label 0. Remember: no coordinate system is right or wrong in itself, some are just more convenient than others. The important thing is to be strictly consistent once you've chosen a coordinate system, and to be mindful of those subtle but crucial minus signs!

3. **Draw free-body diagrams:** Most students find mechanics easier than electromagnetism for the simple reason that mechanics problems are easy to visualize. Free-body diagrams allow you to make the most of this advantage. Make sure you've accounted for all the forces acting on all the bodies in the system. Make ample use of Newton's Third Law, and remember that for systems at rest or at a constant velocity, the net force acting on every body in the system must be zero.

Students too often think that physics problem solving is just a matter of plugging the right numbers into the right equations. The truth is, physics problem solving is more a matter of determining what those right numbers and right equations are. These three steps should help you do just that. Let's look at some mechanical systems.

Pulleys

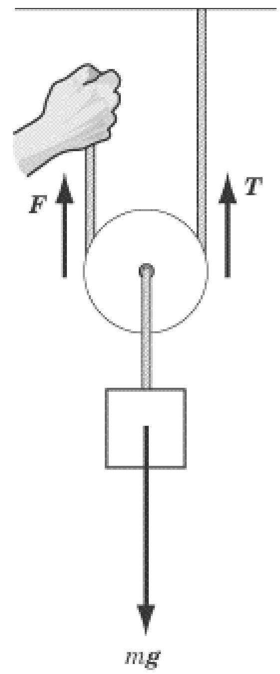


Pulleys are simple machines that consist of a rope that slides around a disk, called a block. Their main function is to change the direction of the tension force in a rope. The pulley systems that appear in introductory physics almost always consist of idealized, massless and frictionless pulleys, and idealized ropes that are massless and that don't stretch. These somewhat unrealistic parameters mean that:

1. The rope slides without any resistance over the pulley, so that the pulley changes the direction of the tension force without changing its magnitude.
2. You can apply the law of conservation of energy to the system without worrying about the energy of the rope and pulley.
3. You don't have to factor in the mass of the pulley or rope when calculating the effect of a force exerted on an object attached to a pulley system.

The one exception to this rule is the occasional problem you might find regarding the torque applied to a pulley block. In such a problem, you will have to take the pulley's mass into account. We'll deal with this special case in a later unit, when we look at torque.

The Purpose of Pulleys

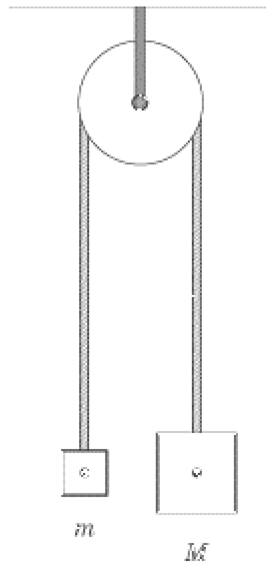


We use pulleys to lift objects because they reduce the amount of force we need to exert. For example, say that you are applying force $\boldsymbol{F}$ to the mass in the figure above. How does $\boldsymbol{F}$ compare to the force you would have to exert in the absence of a pulley?

To lift mass m at a constant velocity without a pulley, you would have to apply a force equal to the mass's weight, or a force of mg upward. Using a pulley, the mass must still be lifted with a force of mg upward, but this force is distributed between the tension of the rope attached to the ceiling, $\boldsymbol{T}$, and the tension of the rope gripped in your hand, $\boldsymbol{F}$.

Because there are two ropes pulling the block, and hence the mass, upward, there are two equal upward forces, $\boldsymbol{F}$ and $\boldsymbol{T}$. We know that the sum of these forces is equal to the gravitational force pulling the mass down, so $F + T = 2F = mg$, or $F = mg/2$. Therefore, you need to pull with only one half the force you would have to use to lift mass m if there were no pulley.

Standard Pulley Problem

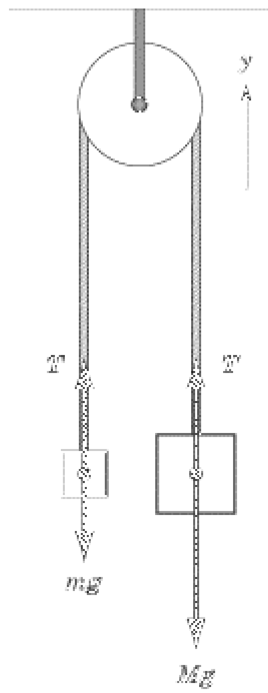


The figure above represents a pulley system where masses *m* and *M* are connected by a rope over a massless and frictionless pulley. Note that $M > m$ and both masses are at the same height above the ground. The system is initially held at rest, and is then released. We will learn to calculate the acceleration of the masses, the velocity of mass *m* when it moves a distance *h*, and the work done by the tension force on mass *m* as it moves a distance *h*.

Before we start calculating values for acceleration, velocity, and work, let's go through the three steps for problem solving:

1. **Ask yourself how the system will move:** From experience, we know that the heavy mass, *M*, will fall, lifting the smaller mass, *m*. Because the masses are connected, we know that the velocity of mass *m* is equal in magnitude to the velocity of mass *M*, but opposite in direction. Likewise, the acceleration of mass *m* is equal in magnitude to the acceleration of mass *M*, but opposite in direction.
2. **Choose a coordinate system:** Some diagrams will provide a coordinate system for you. If they don't, choose one that will simplify your calculations. It is often easiest to choose the direction of the acceleration, regardless of whether it is up, down, left, or right, as positive.

3. **Draw free-body diagrams:** We know that this pulley system will accelerate when released, so we shouldn't expect the net forces acting on the bodies in the system to be zero. Your free-body diagram should end up looking something like the figure below:



Note that the tension force, ***T***, on each of the blocks is of the same magnitude. In any non-stretching rope (the only kind of rope you'll encounter in introductory physics), the tension, as well as the velocity and acceleration, is the same at every point. Now, after preparing ourselves to understand the problem, we can begin answering some questions.

1. What is the acceleration of mass M?
2. What is the velocity of mass m after it travels a distance h?
3. What is the work done by the force of tension in lifting mass m a distance h?

1. What is the acceleration of mass M?

Because the acceleration of the rope is of the same magnitude at every point in the rope, the acceleration of the two masses will also be of equal magnitude. A free body diagram allows you to treat each mass separately, which makes the problem easier. Apply Newton's Second Law to each mass, choosing the direction of the acceleration as positive. When you add up the forces, think of it as "Forces that help the mass accelerate – forces that hold the mass back." Applying this to the diagram, we find:

$$\text{For mass } M\text{: } Mg - T = Ma$$
$$\text{For mass } m\text{: } T - mg = ma$$

By adding the first equation to the second, we find $(M - m)g = (M + m)a$ *or*

$$a = \frac{(M-m)g}{M+M}$$

This result gives us a general formula for the acceleration of any pulley system with unequal masses, M and m. In general for a pulley system:

$$a = \frac{\text{Total Unbalanced Force}}{\text{Total System Mass}}$$

2. What is the velocity of mass m after it travels a distance h?

We could solve this problem by plugging numbers into the kinematics equations, but as you can see, the formula for the acceleration of the pulleys is a bit unwieldy, so the kinematics equations may not be the best approach. Instead, we can tackle this problem in terms of energy. Because the masses in the pulley system are moving up and down, their movement corresponds with a change in gravitational potential energy. Because mechanical energy, E, is conserved, we know that any change in the potential energy, U, of the system will be accompanied by an equal but opposite change in the kinetic energy, K, of the system:

$$\Delta K = -\Delta U$$

Remember that since the system begins at rest, $K_{initial} = 0$ *. As the masses move, mass M loses Mgh joules of potential energy, whereas mass m gains mgh joules of potential energy. Applying the law of conservation of mechanical energy, we find:*

$$\frac{1}{2}Mv^2 + \frac{1}{2}mv^2 = -(mgh - Mgh)$$

$$\frac{1}{2}(M+m)v^2 = (M-m)gh$$

$$v = \sqrt{2gh\left(\frac{M-m}{M+M}\right)}$$

Mass m is moving in the positive y direction.

We admit it: the above formula is pretty scary to look at. But it's less important that you have this exact formula memorized, and more important that you understand the principle by which it was derived. You may find a question that involves a derivation of this or some related formula, so it's good to have at least a rough understanding of the relationship between mass, displacement, and velocity in a pulley system.

3. What is the work done by the force of tension in lifting mass m a distance h?

Since the tension force, ***T****, is in the same direction as the displacement,* ***h****, we know that the work done is equal to hT. But what is the magnitude of the tension force? We know that the sum of forces acting on m is T – mg, which is equal to ma. Therefore, T = m(g – a). From the solution to question 1, we know that a = g(M – m)/(M + m), so substituting in for a, we get:*

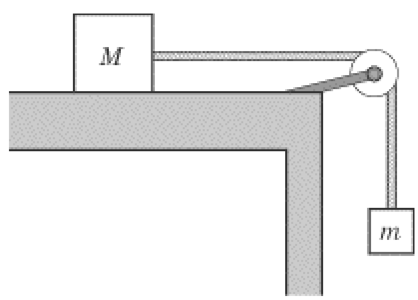
$$W = hT = m(g-a)h$$

$$W = m\left[g - \frac{g(M-m)}{(M+m)}\right]$$

$$W = mgh\left(1 - \frac{M-m}{M+m}\right)$$

A Pulley on a Table

Now imagine that masses *m* and *M* are in the following arrangement:



Let's assume that mass *M* has already begun to slide along the table, and its movement is opposed by the force of kinetic friction, $F_{fr} = \mu_k N$ where μ_k is the coefficient of kinetic friction, and *N* is the normal force acting between the mass and the table. If the mention of friction and normal forces frightens you, you might want to flip back to unit on friction and do a little reviewing.

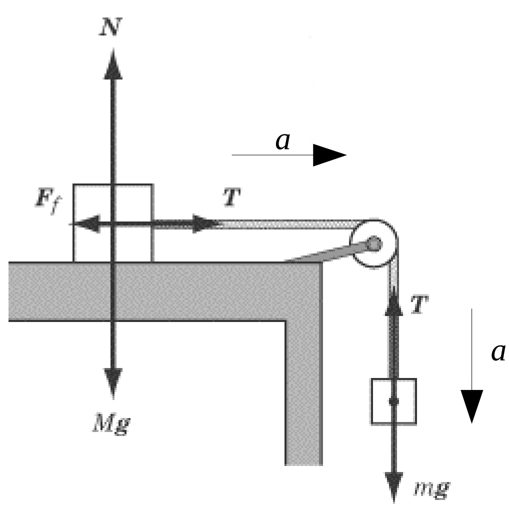
So let’s approach this problem with our handy three-step problem-solving method:

1. **Ask yourself how the system will move:** First, we know that mass m is falling and dragging mass M off the table. The force of kinetic friction opposes the motion of mass M. We also know, since both masses are connected by a non-stretching rope, that the two masses must have the same velocity and the same acceleration.

2. **Choose a coordinate system:** For almost every problem it will be easier if we set our coordinate system relative to the direction of motion rather than worrying about proper x- and y-axes. Make the direction of acceleration positive. This way, when we apply Newton's Second Law to the free body diagram, we can think of it as:

$$\text{(Forces Helping Acceleration)} - \text{(Forces Hindering Acceleration)} = ma$$

3. **Draw free-body diagrams:** The above description of the coordinate system may be a bit confusing. That’s why a diagram can often be a lifesaver.



Given this information, can you calculate the acceleration of the masses? If you think analytically and don’t panic, you can. Since they are attached by a rope, we know that both masses have the same velocity, and hence the same acceleration, a. We also know the net force acting on both masses: the net force acting on mass M is $T - \mu_k mg$, and the net force acting on mass m is $mg - T$.

We can then apply Newton's Second Law to both of the masses, giving us two equations involving a:

For mass M: $T - \mu_k mg = Ma$

For mass m: $mg - T = ma$

Adding the two equations, we find $mg - \mu_k mg = (M + m)\,a$. Solving for a, we get:

$$a = \frac{mg(1 - \mu_k)}{M + m}$$

Notice how the presence of μ_k reduces the acceleration from what it would be in a frictionless situation ($\mu_k = 0$). This is exactly what we expect from friction.

If You Plan to Take SAT II Physics: How These Complex Formulas Will Be Tested

It is highly unlikely that SAT II Physics will ask a question that involves remembering and then plugging numbers into an equation like this one. Remember: SAT II Physics places far less emphasis on math than your high school physics class. The test writers don't want to test your ability to recall a formula or do some simple math. Rather, they want to determine whether you understand the formulas you've memorized. Here are some examples of the kinds of questions you might be asked regarding the pulley system in the free-body diagram above:

1. **Which of the following five formulas represents the acceleration of the pulley system?** You would then be given five different mathematical formulas, one of which is the correct formula. The test writers would not expect you to have memorized the correct formula, but they would expect you to be able to derive it.

2. **Which of the following is a way of maximizing the system's acceleration?** You would then be given options like "maximize M and m and minimize μ," or "maximize μ and m and minimize M." With such a question, you don't even need to know the correct formula, but you do need to understand how the pulley system works. The downward motion is due to the gravitational force on m and is opposed by the force of friction on M, so we would maximize the downward acceleration by maximizing m and minimizing M and μ.

3. **If the system does not move, which of the following must be true?** You would then be given a number of formulas relating M, m, and μ. The idea behind such a question is that the system does not move if the downward force on m is less than or equal to the force of friction on M, so $mg \leq \mu MG$.

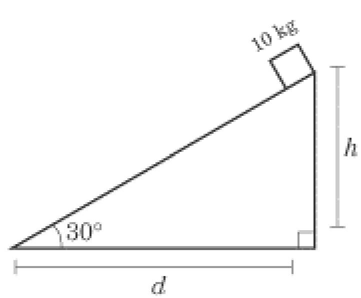
These examples are perhaps less demanding than a question that expects you to derive or recall a complex formula and then plug numbers into it, but they are still difficult questions. In fact, they are about as difficult as mechanics questions on SAT II Physics will get.

Inclined Planes

What we call wedges or slides in everyday language are called inclined planes in physics-speak. From our experience on slides during recess in elementary school, sledding down hills in the winter, and skiing, we know that when people are placed on slippery inclines, they slide down the slope. We also know that slides can sometimes be sticky, so that when you are at the top of the incline, you need to give yourself a push to overcome the force of static friction. As you descend a sticky slide, the force of kinetic friction opposes your motion. In this section, we will consider problems involving inclined planes both with and without friction. Since they're simpler, we'll begin with frictionless planes.

Frictionless Inclined Planes

Suppose you place a 10 kg box on a frictionless 30º inclined plane and release your hold, allowing the box to slide to the ground, a horizontal distance of *d* meters and a vertical distance of *h* meters.

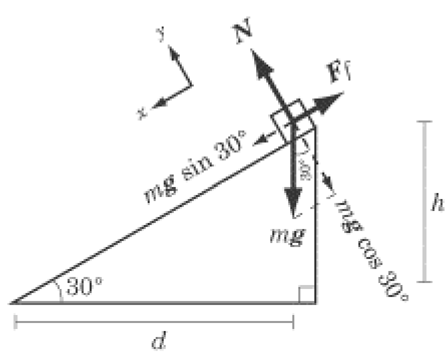


Before we continue, let's follow those three important preliminary steps for solving problems in mechanics:

1. **Ask yourself how the system will move:** Because this is a frictionless plane, there is nothing to stop the box from sliding down to the bottom. Experience suggests that the steeper the incline, the faster an object will slide, so we can expect the acceleration and velocity of the box to be affected by the angle of the plane.

2. **Choose a coordinate system:** Because we're interested in how the box slides along the inclined plane, we would do better to orient our coordinate system to the slope of the plane. The *x*-axis runs parallel to the plane, where downhill is the positive *x* direction, and the *y*-axis runs perpendicular to the plane, where up is the positive *y* direction.

3. **Draw free-body diagrams:** The two forces acting on the box are the force of gravity, acting straight downward, and the normal force, acting perpendicular to the inclined plane, along the *y*-axis. Because we've oriented our coordinate system to the slope of the plane, we'll have to resolve the vector for the gravitational force, ***mg***, into its *x*- and *y*-components. If you recall what we learned about vector decomposition in the first unit, you'll know you can break ***mg*** down into a vector of magnitude cos 30º in the negative *y* direction and a vector of magnitude sin 30º in the positive *x* direction. The result is a free-body diagram that looks something like this:



Decomposing the $m\mathbf{g}$ vector gives a total of three force vectors at work in this diagram: the *y*-component of the gravitational force and the normal force, which cancel out; and the *x*-component of the gravitational force, which pulls the box down the slope. Note that the steeper the slope, the greater the force pulling the box down the slope.

Now let's solve some problems. For the purposes of these problems, take the acceleration due to gravity to be $g = 10\ \text{m/s}^2$. We will give you the values of the relevant trigonometric functions: $\cos 30^\circ = \sin 60^\circ = 0.866$, $\cos 60^\circ = \sin 30^\circ = 0.500$.

1. What is the magnitude of the normal force?
2. What is the acceleration of the box?
3. What is the velocity of the box when it reaches the bottom of the slope?
4. What is the work done on the box by the force of gravity in bringing it to the bottom of the plane?

1. What is the magnitude of the normal force?

The box is not moving in the y direction, so the normal force must be equal to the y-component of the gravitational force. Calculating the normal force is then just a matter of plugging a few numbers in for variables in order to find the y-component of the gravitational force:

$$\begin{aligned} N &= mg\cos 30^\circ \\ &= (10\ \text{kg})(10\ \text{m/s}^2)(0.866) \\ &= 86.6\ \text{N} \end{aligned}$$

2. What is the acceleration of the box?

We know that the force pulling the box in the positive x direction has a magnitude of $mg\sin 30^\circ$. Using Newton's Second Law, F = ma, we just need to solve for a:

$$\begin{aligned} ma &= mg\sin 30^\circ \\ a &= g\sin 30^\circ \\ &= (10\ \text{m/s}^2)(0.500) \\ &= 5.00\ \text{m/s}^2 \end{aligned}$$

3. What is the velocity of the box when it reaches the bottom of the slope?

Because we're dealing with a frictionless plane, the system is closed and we can invoke the law of conservation of mechanical energy. At the top of the inclined plane, the box will not be moving and so it will have an initial kinetic energy of zero ($K_{initial} = 0$). Because it is a height h above the bottom of the plane, it will have a gravitational potential energy of U = mgh. Adding kinetic and potential energy, we find that the mechanical energy of the system is:

$$E = K + U = 0 + mgh = mgh$$

At the bottom of the slope, all the box's potential energy will have been converted into kinetic energy. In other words, the kinetic energy, ½ mv^2, of the box at the bottom of the slope is equal to the potential energy, mgh, of the box at the top of the slope. Solving for v, we get:

$$\begin{aligned} v &= \sqrt{2gh} \\ &= \sqrt{2(10\ \text{m/s}^2)h} \\ &= 4.47\sqrt{h} \end{aligned}$$

4. What is the work done on the box by the force of gravity in bringing it to the bottom of the inclined plane?

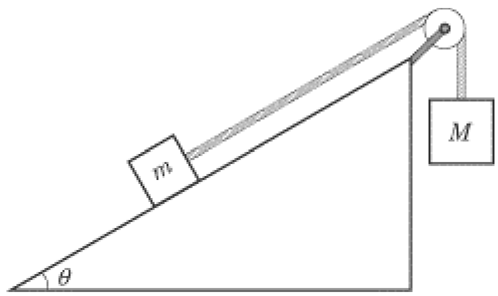
The fastest way to solve this problem is to appeal to the work-energy theorem, which tells us that the work done on an object is equal to its change in kinetic energy. At the top of the slope the box has no kinetic energy, and at the bottom of the slope its kinetic energy is equal to its potential energy at the top of the slope, mgh. So the work done on the box is:

$$\begin{aligned} W &= mgh = (10\ \text{kg})(10\ \text{m/s}^2)\,h \\ &= 100\,h\ \text{J} \end{aligned}$$

Note that the work done is independent of how steep the inclined plane is, and is only dependent on the object's change in height when it slides down the plane.

Frictionless Inclined Planes with Pulleys

Let's bring together what we've learned about frictionless inclined planes and pulleys on tables into one exciting über-problem:

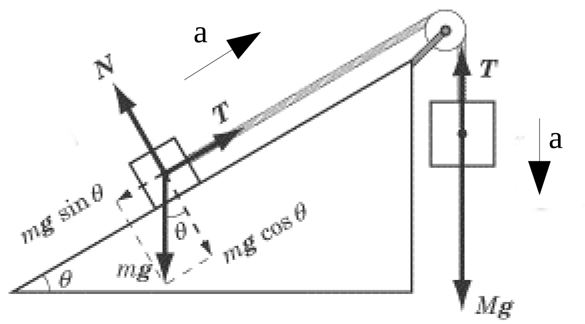


Assume for this problem that $Mg > mg\sin\theta$; that is, mass M will pull mass m up the slope. Now let's ask those three all-important preliminary questions:

1. **Ask yourself how the system will move:** Because the two masses are connected by a rope, we know that they will have the same velocity and acceleration. We also know that the tension in the rope is constant throughout its length. Because $Mg > mg\sin\theta$, we know that when the system is released from rest, mass M will move downward and mass m will slide up the inclined plane.

2. **Choose a coordinate system:** Do the same thing here that we did with the previous pulley-on-a-table problem. Make the x-axis parallel to the rope, with the positive x direction being up for mass M and downhill for mass m, and the negative x direction being down for mass M and uphill for mass m. Make the y-axis perpendicular to the rope, with the positive y-axis being away from the inclined plane, and the negative y-axis being toward the inclined plane.

3. **Draw free-body diagrams:** We've seen how to draw free-body diagrams for masses suspended from pulleys, and we've seen how to draw free-body diagrams for masses on inclined planes. All we need to do now is synthesize what we already know.



Now let's tackle a couple of questions:

1. What is the acceleration of the masses?
2. What is the velocity of mass *m* after mass *M* has fallen a distance *h*?

1. What is the acceleration of the masses?

First, let's determine the net force acting on each of the masses. Applying Newton's Second Law we get:

For mass M: $Mg - T = Ma$

For mass m: $T - mg \sin\theta = ma$

Adding these two equations together, we find that $Mg - mg \sin\theta = (M + m)a$ *. Solving for a, we get:*

$$a = g\left(\frac{M - m\sin\theta}{M + m}\right)$$

Because $M > m \sin\theta$*, the acceleration is positive, meaning that it is in the direction that we drew it.*

2. What is the velocity of mass m after mass M has fallen a distance h?

Once again, the inclined plane is frictionless, so we are dealing with a closed system and we can apply the law of conservation of mechanical energy. Since the masses are initially at rest, . $K_{initial} = 0$. Since mass M falls a distance h, its potential energy changes by –Mgh. If mass M falls a distance h, then mass m must slide the same distance up the slope of the inclined plane, or a vertical distance of $h \sin\theta$. Therefore, mass m's potential energy increases by $mgh \sin\theta$. Because the sum of potential energy and kinetic energy cannot change, we know that the kinetic energy of the two masses increases precisely by the amount that their potential energy decreases. We have all we need to scribble out some equations and solve for v:

$$\Delta K + \Delta U = 0$$

$$\left(\frac{1}{2} M v^2 + \frac{1}{2} m v^2\right) + \left(-Mgh + mgh \sin\theta\right) = 0$$

$$\frac{1}{2}(M+m) v^2 = gh(M - m \sin\theta)$$

$$v = \sqrt{\frac{2gh(M - m\sin\theta)}{m+M}}$$

Finally, note that the velocity of mass m is in the uphill direction.

As with the complex equations we encountered with pulley systems above, you needn't trouble yourself with memorizing a formula like this. If you understand the principles at work in this problem and would feel somewhat comfortable deriving this formula, you know more than the SAT II Physics test will likely ask of you.

Inclined Planes With Friction

There are two significant differences between frictionless inclined plane problems and inclined plane problems where friction is a factor:

1. **There's an extra force to deal with.** The force of friction will oppose the downhill component of the gravitational force.

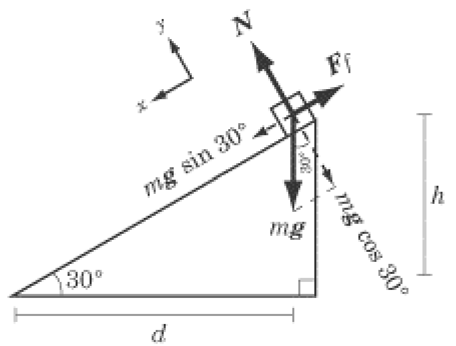
2. **We can no longer rely on the law of conservation of mechanical energy.** Because energy is being lost through the friction between the mass and the inclined plane, we are no longer dealing with a closed system. Mechanical energy is not conserved.

Consider the 10 kg box we encountered in our example of a frictionless inclined plane. This time, though, the inclined plane has a coefficient of kinetic friction of $\mu_k = 0.500$. How will this additional factor affect us? Let's follow three familiar steps:

1. **Ask yourself how the system will move:** If the force of gravity is strong enough to overcome the force of friction, the box will accelerate down the plane. However, because there is a force acting against the box's descent, we should expect it to slide with a lesser velocity than it did in the example of the frictionless plane.

2. **Choose a coordinate system:** There's no reason not to hold onto the co-ordinate system we used before: the positive *x* direction is down the slope, and the positive *y* direction is upward, perpendicular to the slope.

3. **Draw free-body diagrams:** The free-body diagram will be identical to the one we drew in the example of the frictionless plane, except we will have a vector for the force of friction in the negative *x* direction.



Now let's ask some questions about the motion of the box.

1. What is the force of kinetic friction acting on the box?
2. What is the acceleration of the box?
3. What is the work done on the box by the force of kinetic friction?

1. What is the force of kinetic friction acting on the box?

The normal force acting on the box is 86.6 N, exactly the same as for the frictionless inclined plane. The force of kinetic friction is defined as $F_f = \mu_k N$, so plugging in the appropriate values for μ_k and N:

$$F_f = \mu_k N = (0.500)(86.6 \text{ N}) = 43.3 \text{ N}$$

Remember, though, that the force of friction is exerted in the negative x direction, so the correct answer is -43.3 N.

2. What is the acceleration of the box?

The net force acting on the box is the difference between the downhill gravitational force and the force of friction: $F = mg\sin 30^o - F_f$. Using Newton's Second Law, we can determine the net force acting on the box, and then solve for a:

$$ma = mg\sin 30^0 - F_f$$

$$(10 \text{ kg})a = (10 \text{ kg})(10 \text{ m/s}^2)(0.500) - 43.3 \text{ N}$$

$$a = \frac{50 \text{ N} - 43.3 \text{ N}}{10 \text{ kg}} = 0.67 \text{ m/s}^2$$

3. What is the work done on the box by the force of kinetic friction?

*Since **W** = **F** · **s**, the work done by the force of friction is the product of the force of friction and the displacement of the box in the direction that the force is exerted. Because the force of friction is exerted in the negative x direction, we need to find the displacement of the box in the x direction. We know that it has traveled a horizontal distance of d and a vertical distance of h. The Pythagorean Theorem then tells us that the displacement of the box is* $s = \sqrt{d^2 + h^2}$ *. Recalling that the force of friction is – 43.3 N, we know that the work done by the force of friction is*

$$W = -43.3\sqrt{d^2 + h^2}\ \mathrm{J}$$

Note that the amount of work done is negative, because the force of friction acts in the opposite direction of the displacement of the box.

Springs

Questions about springs in physics are usually simple matters of a mass on a spring oscillating back and forth. However, spring motion is the most interesting of the four topics we will cover here because of its generality. The harmonic motion that springs exhibit applies equally to objects moving in a circular path and to the various wave phenomena that we'll study later in this book. So before we dig in to the nitty-gritty of your typical physics spring question, let's look at some general features of harmonic motion.

Oscillation and Harmonic Motion

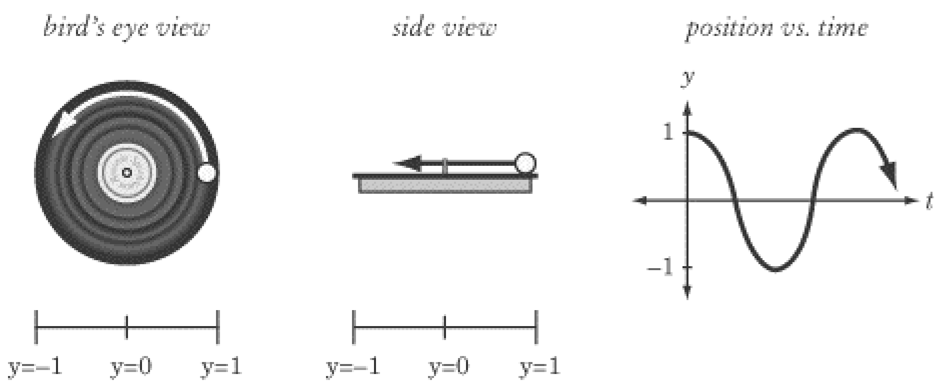
Consider the following physical phenomena:

- When you drop a rock into a still pond, the rock makes a big splash, which causes ripples to spread out to the edges of the pond.
- When you pluck a guitar string, the string vibrates back and forth.
- When you rock a small boat, it wobbles to and fro in the water before coming to rest again.
- When you stretch out a spring and release it, the spring goes back and forth between being compressed and being stretched out.

There are just a few examples of the widespread phenomenon of oscillation. Oscillation is the natural world's way of returning a system to its equilibrium position, the stable position of the system where the net force acting on it is zero. If you throw a system off-balance, it doesn't simply return to the way it was; it oscillates back and forth about the equilibrium position.

A system oscillates as a way of giving off energy. A system that is thrown off-kilter has more energy than a system in its equilibrium position. To take the simple example of a spring, a stretched-out spring will start to move as soon as you let go of it: that motion is evidence of kinetic energy that the spring lacks in its equilibrium position. Because of the law of conservation of energy, a stretched-out spring cannot simply return to its equilibrium position; it must release some energy in order to do so. Usually, this energy is released as thermal energy caused by friction, but there are plenty of interesting exceptions. For instance, a plucked guitar string releases sound energy: the music we hear is the result of the string returning to its equilibrium position.

The movement of an oscillating body is called harmonic motion. If you were to graph the position, velocity, or acceleration of an oscillating body against time, the result would be a sinusoidal wave; that is, some variation of a $y = a \sin bx$ or a $y = a \cos bx$ graph. This generalized form of harmonic motion applies not only to springs and guitar strings, but to anything that moves in a cycle. Imagine placing a pebble on the edge of a turntable, and watching the turntable rotate while looking at it from the side. You will see the pebble moving back and forth in one dimension. The pebble will appear to oscillate just like a spring: it will appear to move fastest at the middle of its trajectory and slow to a halt and reverse direction as it reaches the edge of its trajectory.

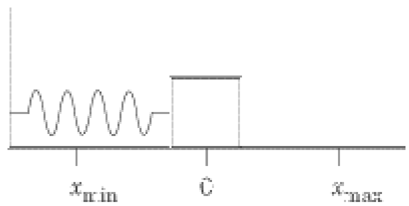


This example serves two purposes. First, it shows you that the oscillation of springs is just one of a wide range of phenomena exhibiting harmonic motion. Anything that moves in a cyclic pattern exhibits harmonic motion. This includes the light and sound waves without which we would have a lot of trouble moving about in the world. Second, we bring it up because the SAT II Physics subject test has been known to test students on the nature of the horizontal or vertical component of the motion of an object in circular motion. As you can see, circular motion viewed in one dimension is harmonic motion.

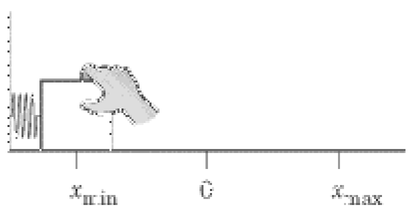
Though harmonic motion is one of the most widespread and important of physical phenomena, your understanding of it will not be taxed to any great extent in introductory physics. In fact, beyond the motion of springs and pendulums, everything you will need to know will be covered in the unit on Waves. The above discussion is mostly meant to fit your understanding of the oscillation of springs into a wider context.

The Oscillation of a Spring

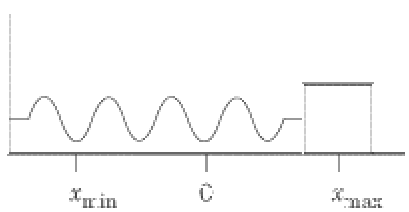
Now let's focus on the harmonic motion exhibited by a spring. To start with, we'll imagine a mass, *m*, placed on a frictionless surface, and attached to a wall by a spring. In its equilibrium position, where no forces act upon it, the mass is at rest. Let's label this equilibrium position $x = 0$. Intuitively, you know that if you compress or stretch out the spring it will begin to oscillate.



Suppose you push the mass toward the wall, compressing the spring, until the mass is in position $x = x_{min}$.



When you release the mass, the spring will exert a force, pushing the mass back until it reaches position $x = x_{max}$, which is called the amplitude of the spring's motion, or the maximum displacement of the oscillator. Note that $x_{min} = -x_{max}$.



By that point, the spring will be stretched out, and will be exerting a force to pull the mass back in toward the wall. Because we are dealing with an idealized frictionless surface, the mass will not be slowed by the force of friction, and will oscillate back and forth repeatedly between x_{min} and x_{max}.

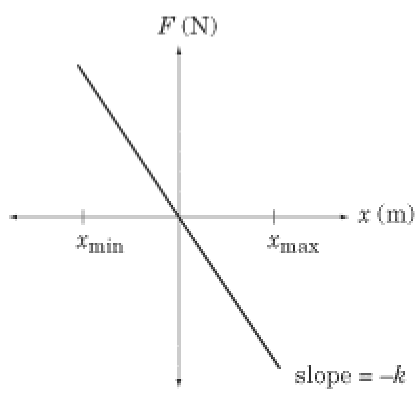
Hooke's Law

This is all well and good, but we can't get very far in sorting out the amplitude, the velocity, the energy, or anything else about the mass's motion if we don't understand the manner in which the spring exerts a force on the mass attached to it. The force, ***F***, that the spring exerts on the mass is defined by Hooke's Law:

$$\boldsymbol{F} = -k\boldsymbol{x}$$

where x is the spring's displacement from its equilibrium position and k is a constant of proportionality called the spring constant. The spring constant is a measure of "springiness": a greater value for k signifies a "tighter" spring, one that is more resistant to being stretched.

Hooke's Law tells us that the further the spring is displaced from its equilibrium position ($\boldsymbol{x}$) the greater the force the spring will exert in the direction of its equilibrium position ($\boldsymbol{F}$). We call $\boldsymbol{F}$ a restoring force: it is always directed toward equilibrium. Because F and x are directly proportional, a graph of F vs. x is a line with slope $-k$.



Simple Harmonic Oscillation

A mass oscillating on a spring is one example of a simple harmonic oscillator. Specifically, a simple harmonic oscillator is any object that moves about a stable equilibrium point and experiences a restoring force proportional to the oscillator's displacement.

For an oscillating spring, the restoring force, and consequently the acceleration, are greatest and positive at x_{min}. These quantities decrease as x approaches the equilibrium position and are zero at $x = 0$. The restoring force and acceleration—which are now negative—increase in magnitude as x approaches x_{max} and are maximally negative at x_{max}.

Important Properties of a Mass on a Spring

There are a number of important properties related to the motion of a mass on a spring, all of which are fair game on the SAT II Physics test. Remember, though: the test makers have no interest in testing your ability to recall complex formulas and perform difficult mathematical operations. You may be called upon to know the simpler of these formulas, but not the complex ones. As we mentioned at the end of the section on pulleys, it's less important that you memorize the formulas and more important that you understand what they mean. If you understand the principle, there probably won't be any questions that will stump you.

Period of Oscillation

The period of oscillation, T, of a spring is the amount of time it takes for a spring to complete a round-trip or cycle. Mathematically, the period of oscillation of a simple harmonic oscillator described by Hooke's Law is:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

This equation tells us that as the mass of the block, m, increases and the spring constant, k, decreases, the period increases. In other words, a heavy mass attached to an easily stretched spring will oscillate back and forth very slowly, while a light mass attached to a resistant spring will oscillate back and forth very quickly.

Frequency

The frequency of the spring's motion tells us how quickly the object is oscillating, or how many cycles it completes in a given time frame. Frequency is inversely proportional to period:

$$f = \frac{1}{T}$$

Frequency is given in units of cycles per second, or hertz (Hz).

Potential Energy

The potential energy of a spring (U_s) is sometimes called elastic energy, because it results from the spring being stretched or compressed. Mathematically, U_s is defined by:

$$U_s = \frac{1}{2}kx^2$$

The potential energy of a spring is greatest when the coil is maximally compressed or stretched, and is zero at the equilibrium position.

Kinetic Energy

You will not be tested on the motion of springs involving friction, so for the purposes of the test the mechanical energy of a spring is a conserved quantity. As we recall, mechanical energy is the sum of the kinetic energy and potential energy.

At the points of maximum compression and extension, the velocity, and hence the kinetic energy, is zero and the mechanical energy is equal to the potential energy, $U_s = ½\, kx^2$.

At the equilibrium position, the potential energy is zero, and the velocity and kinetic energy are maximized. The kinetic energy at the equilibrium position is equal to the mechanical energy:

$$K_{max} = \frac{1}{2}mv^2_{max} = \frac{1}{2}kA^2$$

where A is the amplitude of motion (the maximum displacement from equilibrium).

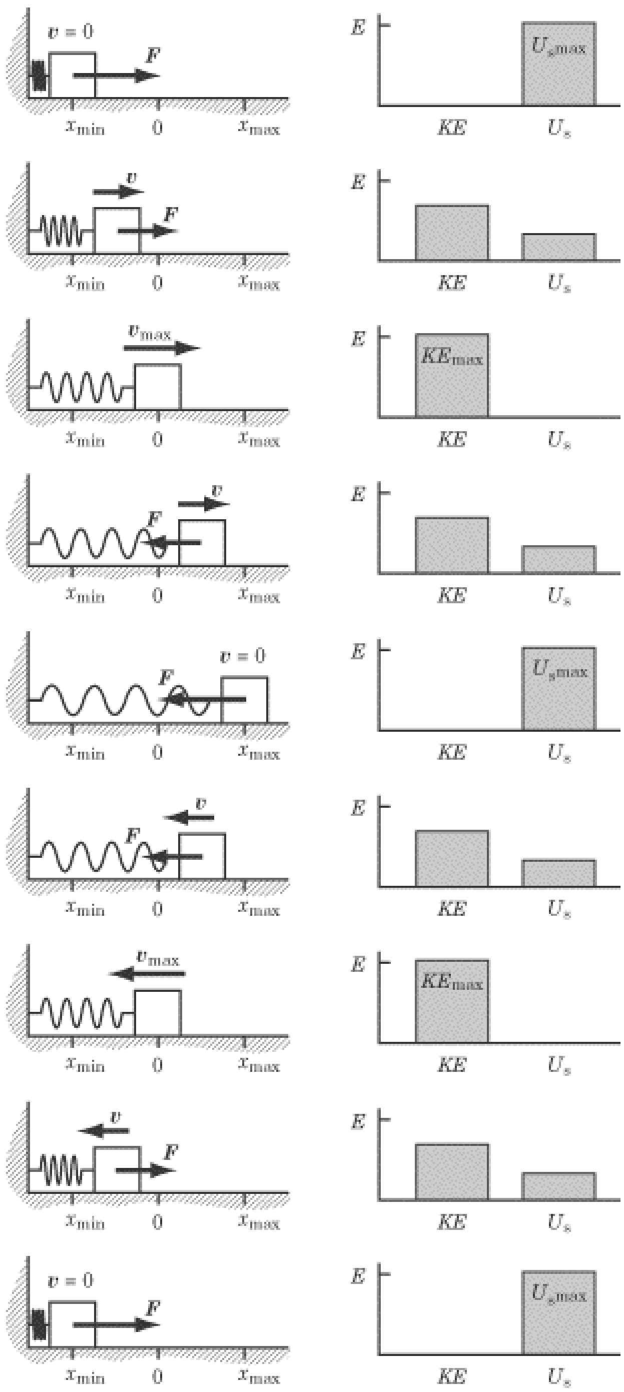
From this equation, we can derive the maximum velocity:

$$v_{max} = A\sqrt{\frac{k}{m}}$$

You won't need to know this equation, but it might be valuable to note that the velocity increases with a large displacement, a resistant spring, and a small mass.

Summary

It is highly unlikely that the formulas discussed above will appear on SAT II Physics. More likely, you will be asked conceptual questions such as: at what point in a spring's oscillation is the kinetic or potential energy maximized or minimized, for instance. The figure on the next page summarizes and clarifies some qualitative aspects of simple harmonic oscillation. Your qualitative understanding of the relationship between force, velocity, and kinetic and potential energy in a spring system is far more likely to be tested than your knowledge of the formulas discussed above.



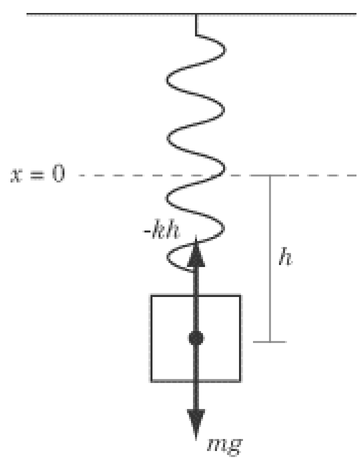
In this figure, ***v*** represents velocity, ***F*** represents force, *KE* represents kinetic energy, and U_s represents potential energy.

Vertical Oscillation of Springs

Now let's consider a mass attached to a spring that is suspended from the ceiling. Questions of this sort have a nasty habit of coming up on SAT II Physics. The oscillation of the spring when compressed or extended won't be any different, but we now have to take gravity into account.

Equilibrium Position

Because the mass will exert a gravitational force to stretch the spring downward a bit, the equilibrium position will no longer be at $x = 0$, but at $x = –h$, where h is the vertical displacement of the spring due to the gravitational pull exerted on the mass. The equilibrium position is the point where the net force acting on the mass is zero; in other words, the point where the upward restoring force of the spring is equal to the downward gravitational force of the mass.



Combining the restoring force, $F = –kx$, and the gravitational force, $F = mg$, we can solve for h:

$$-kx = -k(-h) = mg$$

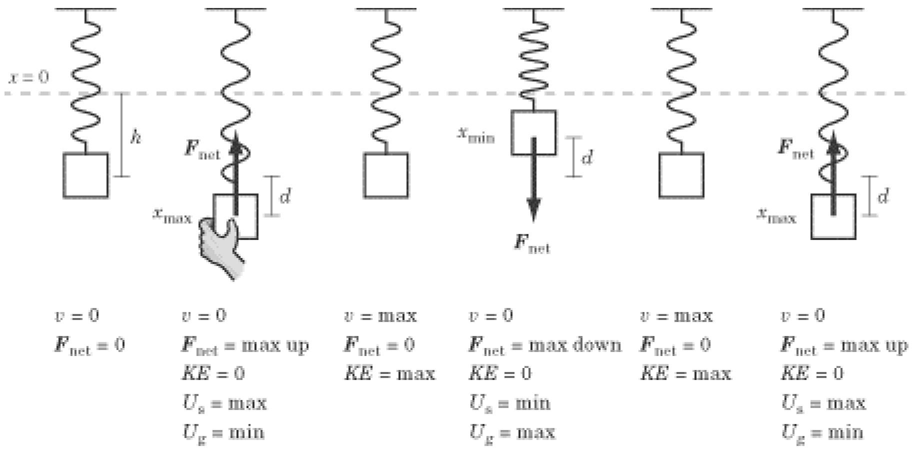
$$kh = mg$$

$$h = \frac{mg}{k}$$

Since m is in the numerator and k in the denominator of the fraction, the mass displaces itself more if it has a large weight and is suspended from a lax spring, as intuition suggests.

A Vertical Spring in Motion

If the spring is then stretched a distance d, where $d < h$, it will oscillate between $x_{max} = -h - d$ and $x_{min} = -h + d$.



Throughout the motion of the mass, the force of gravity is constant and downward. The restoring force of the spring is always upward, because even at x_{min} the mass is below the spring's initial equilibrium position of $x = 0$. Note that if d were greater than h, x_{min} would be above $x = 0$, and the restoring force would act in the downward direction until the mass descended once more below $x = 0$.

According to Hooke's Law, the restoring force decreases in magnitude as the spring is compressed. Consequently, the net force downward is greatest at $x = x_{min}$ and the net force upward is greatest at $x = x_{max}$.

Energy

The mechanical energy of the vertically oscillating spring is:

$$E = K + U_g - U_s$$

where U_g is gravitational potential energy and U_s is the spring's elastic potential energy.

Note that the velocity of the block is zero at $x = x_{min}$ and $x = x_{max}$, and maximized at the equilibrium position, $x = -h$. Consequently, the kinetic energy of the spring is zero for $x = x_{max}$ and $x = x_{min}$ and is greatest at $x = -h$. The gravitational potential energy of the system increases with the height of the mass. The elastic potential energy of the spring is greatest when the spring is maximally extended at x_{max} and decreases with the extension of the spring.

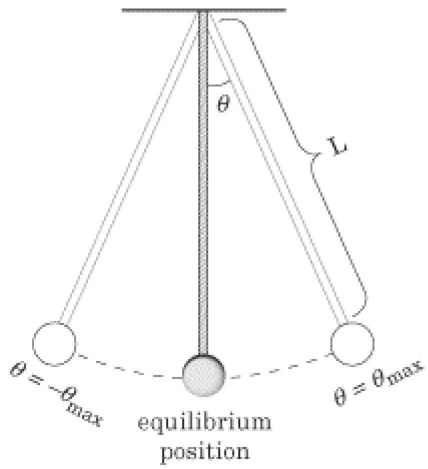
How This Knowledge Will Be Tested on the SAT II

Most of the questions on SAT II Physics that deal with spring motion will ask qualitatively about the energy or velocity of a vertically oscillating spring. For instance, you may be shown a diagram capturing one moment in a spring's trajectory and asked about the relative magnitudes of the gravitational and elastic potential energies and kinetic energy. Or you may be asked at what point in a spring's trajectory the velocity is maximized. The answer, of course, is that it is maximized at the equilibrium position. It is far less likely that you will be asked a question that involves any sort of calculation.

Pendulums

A pendulum is defined as a mass, or bob, connected to a rod or rope, that experiences simple harmonic motion as it swings back and forth without friction. The equilibrium position of the pendulum is the position when the mass is hanging directly downward.

Consider a pendulum bob connected to a massless rope or rod that is held at an angle θ_{max} from the horizontal. If you release the mass, then the system will swing to position $-\theta_{max}$ and back again.

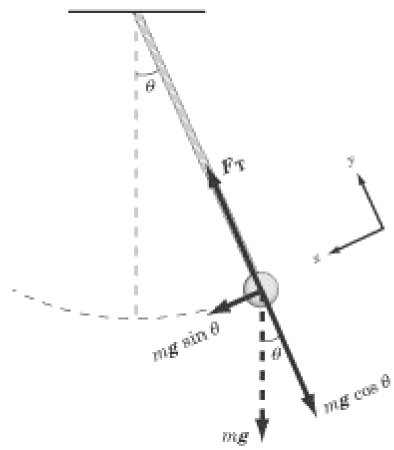


The oscillation of a pendulum is much like that of a mass on a spring. However, there are significant differences, and many a student has been tripped up by trying to apply the principles of a spring's motion to pendulum motion.

Properties of Pendulum Motion

As with springs, there are a number of properties of pendulum motion that you might be tested on, from frequency and period to kinetic and potential energy. Let's apply our three-step method of approaching special problems in mechanics and then look at the formulas for some of those properties:

1. **Ask yourself how the system will move:** It doesn't take a rocket scientist to surmise that when you release the pendulum bob it will accelerate toward the equilibrium position. As it passes through the equilibrium position, it will slow down until it reaches position $-\theta_{max}$, and then accelerate back. At any given moment, the velocity of the pendulum bob will be perpendicular to the rope. The pendulum's trajectory describes an arc of a circle, where the rope is a radius of the circle and the bob's velocity is a line tangent to the circle.
2. **Choose a coordinate system:** We want to calculate the forces acting on the pendulum at any given point in its trajectory. It will be most convenient to choose a *y*-axis that runs parallel to the rope. The *x*-axis then runs parallel to the instantaneous velocity of the bob so that, at any given moment, the bob is moving along the *x*-axis.
3. **Draw free-body diagrams:** Two forces act on the bob: the force of gravity, $\boldsymbol{F} = m\boldsymbol{g}$, pulling the bob straight downward and the tension of the rope, $\boldsymbol{F}_T$, pulling the bob upward along the *y*-axis. The gravitational force can be broken down into an *x*-component, $mg\sin\theta$, and a *y*-component, $mg\cos\theta$. The *y*-component balances out the force of tension—the pendulum bob doesn't accelerate along the *y*-axis—so the tension in the rope must also be $mg\cos\theta$. Therefore, the tension force is maximum for the equilibrium position and decreases with θ. The restoring force is $mg\sin\theta$, so, as we might expect, the restoring force is greatest at the endpoints of the oscillation, $\theta = \pm\,\theta_{max}$ and is zero when the pendulum passes through its equilibrium position.



You'll notice that the restoring force for the pendulum, $mg\sin\theta$, is not directly proportional to the displacement of the pendulum bob, θ, which makes calculating the various properties of the pendulum very difficult. Fortunately, pendulums usually only oscillate at small angles, where $\sin\theta \approx \theta$ (when measured in radians). In such cases, we can derive more straightforward formulas, which are admittedly only approximations. However, they're good enough for the purposes of an introductory physics course.

Period

The period of oscillation of the pendulum, T, is defined in terms of the acceleration due to gravity, g, and the length of the pendulum, L:

$$T = 2\pi\sqrt{\frac{L}{g}}$$

This is a pretty scary-looking equation, but there's really only one thing you need to gather from it: the longer the pendulum rope, the longer it will take for the pendulum to oscillate back and forth. You should also note that the mass of the pendulum bob and the angle of displacement play no role in determining the period of oscillation.

If the pendulum happens to be on Earth, we can take advantage of an neat coincidence concerning g:

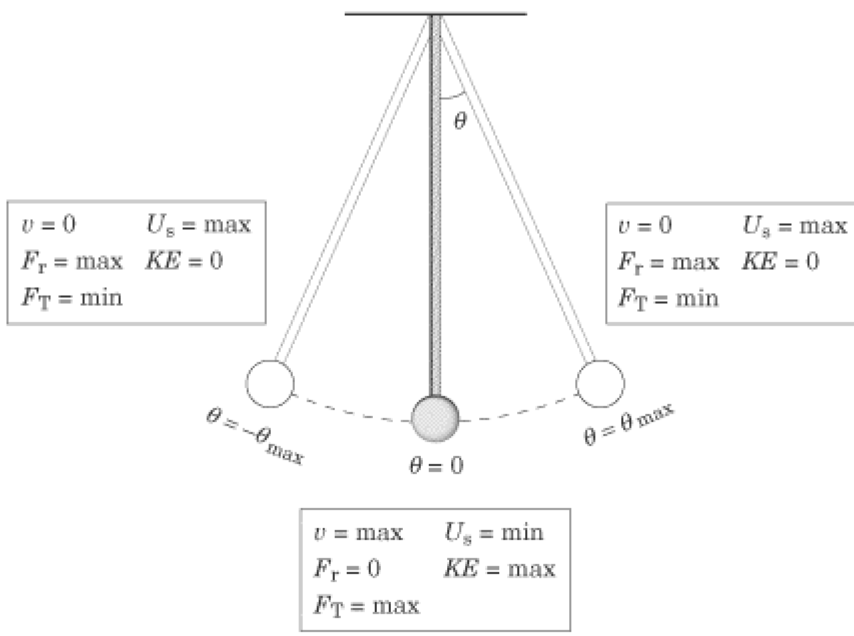
$$\sqrt{g} = \sqrt{9.81} = 3.132 \approx \pi$$

and the above equation reduces to $T = 2\sqrt{L}$. Remember, this is <u>only</u> valid on Earth.

Energy

The mechanical energy of the pendulum is a conserved quantity. The potential energy of the pendulum, *mgh*, increases with the height of the bob; therefore the potential energy is minimized at the equilibrium point and is maximized at $\theta = \pm\,\theta_{max}$. Conversely, the kinetic energy and velocity of the pendulum are maximized at the equilibrium point and minimized when $\theta = \pm\,\theta_{max}$.

The figure below summarizes this information in a qualitative manner, which is the manner in which you are most likely to find it on a test. In this figure, v signifies velocity, F_r signifies the restoring force, F_T signifies the tension in the pendulum string, U_g signifies potential energy, and KE signifies kinetic energy.

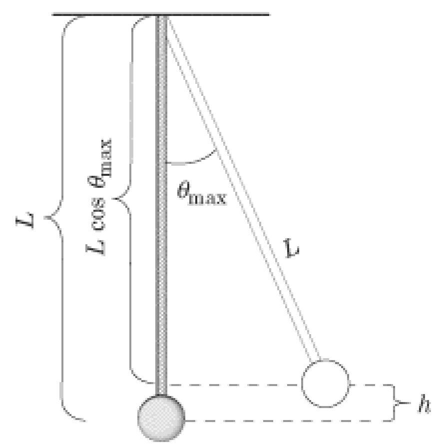


Velocity

Calculating the velocity of the pendulum bob at the equilibrium position requires that we arrange our coordinate system so that the height of the bob at the equilibrium position is zero. Then the total mechanical energy is equal to the kinetic energy at the equilibrium point where $U_g = 0$. The total mechanical energy is also equal to the total potential energy at $\pm\,\theta_{max}$ where $K = 0$. Putting these equalities together, we get

$$E \;=\; \frac{1}{2}mv_{max}^2 \;=\; mgh$$

But what is *h*?



From the figure, we see that $h = L - L\cos\theta_{max}$. If we plug that value into the equation above, we can solve for *v*:

$$\frac{1}{2}mv_{max}^2 = mgL\left(1-\cos\theta_{max}\right)$$

$$v_{max} \;=\; \sqrt{2gL\left(1-\cos\theta_{max}\right)}$$

Don’t let a big equation frighten you. Just register what it conveys: the longer the string and the greater the angle, the faster the pendulum bob will move.

How This Knowledge Will Be Tested

Again, don't worry too much about memorizing equations: most of the questions on pendulum motion will be qualitative. There may be a question asking you at what point the tension in the rope is greatest (at the equilibrium position) or where the bob's potential energy is maximized (at $\theta = \pm \theta_{max}$). It's highly unlikely that you'll be asked to give a specific number.

Key Formulas

Hooke's Law	$\boldsymbol{F} = -kx$
Period of Oscillation of a Spring	$T = 2\pi\sqrt{\frac{m}{k}}$
Frequency	$f = \frac{1}{T}$
Potential Energy of a Spring	$U_s = \frac{1}{2}kx^2$
Velocity of a Spring at Equilibrium Position	$v_{max} = A\sqrt{\frac{k}{m}}$
Period of Oscillation of a Pendulum	$T = 2\pi\sqrt{\frac{L}{g}}$
Velocity of a Pendulum at Equilibrium Position	$v_{max} = \sqrt{2gL(1-\cos\theta_{max})}$